Uitwerking tentamen Golven en Optica 21/11/96 (zonder garantie)

1a. $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$ $\quad \frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \frac{k}{m}x = 0$ met $\gamma = \frac{b}{m}$, $\omega_0^2 = \frac{k}{m}$

$\gamma = \frac{30}{25} = 1.2\ s^{-1}$ $\quad \omega_0 = \sqrt{\frac{75}{25}} = \sqrt{3}$ Hz $\quad Q = \frac{\omega_0}{\gamma} = \frac{\sqrt{3}}{1.2} = 1.44$

b. $x = Ce^{qt} \Rightarrow q^2 + \gamma q + \omega_0^2 = 0 \Rightarrow q = -\frac{1}{2}\gamma \pm \sqrt{\frac{1}{4}\gamma^2 - \omega_0^2}$ Hier $\omega_0^2 > \frac{1}{4}\gamma^2$

$\Rightarrow q = -\frac{1}{2}\gamma \pm i\omega$ met $\omega = \sqrt{\omega_0^2 - \frac{1}{4}\gamma^2} \Rightarrow x = C_1 e^{-\frac{\gamma}{2}t}e^{i\omega t} + C_2 e^{-\frac{\gamma}{2}t}e^{-i\omega t}$

eis x reëel $\Rightarrow C_2 = C_1^* \Rightarrow x = e^{-\frac{1}{2}\gamma t}\, 2\,\mathrm{Re}(C_1 e^{i\omega t}) = A e^{-\frac{1}{2}\gamma t}\cos(\omega t + \alpha)$

$x(0) = x_0$, $\dot{x}(0) = 0$ $\quad A\cos\alpha = x_0$ $\quad \dot{x}(0) = \left[-A\frac{1}{2}\gamma e^{-\frac{1}{2}\gamma t}\cos(\omega t + \alpha) - A\omega e^{-\frac{1}{2}\gamma t}\sin(\omega t + \alpha)\right]_{t=0} =$

$= -A\frac{1}{2}\gamma\cos\alpha - A\omega\sin\alpha = 0 \Rightarrow \tan\alpha = -\frac{\gamma}{2\omega} \rightarrow \frac{A}{x_0} = \frac{1}{\cos\alpha} = \sqrt{\tan^2\alpha + 1} =$

$= \sqrt{\frac{\gamma^2}{4\omega^2} + 1} = \frac{1}{\omega}\sqrt{\frac{1}{4}\gamma^2 + \omega^2} = \frac{\omega_0}{\omega} \Rightarrow A = x_0\frac{\omega_0}{\omega}$ $\quad x = x_0\frac{\omega_0}{\omega}e^{-\frac{1}{2}\gamma t}\cos(\omega t + \alpha)$

$\omega = \sqrt{\omega_0^2 - \frac{1}{4}\gamma^2} = 1.62$ Hz $\quad A = 0.1\frac{\sqrt{3}}{1.62} = 0.11$ m $\quad \alpha = \arctan\left(\frac{-1.2}{2\cdot 1.62}\right) = -0.35$ rad

$x = 0.11\, e^{-0.6t}\cos(1.62t - 0.35)$

c. Licht gedempt: $\omega_0^2 > \frac{1}{4}\left(\frac{b}{m}\right)^2 \Rightarrow b < (4m^2\omega_0^2)^{\frac{1}{2}} = 86.6\ \mathrm{Nm^{-1}s}$

Kritisch gedempt $b = 86.6\ \mathrm{Nm^{-1}s}$, overgedempt $b > 86.6\ \mathrm{Nm^{-1}s}$

d. Positie langs veer s ($s=0$ vast, $s=\ell$ m) $\Rightarrow$ massa elementje $ds = \frac{M}{\ell}ds$

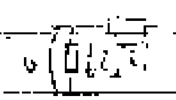
snelheid $v(s) = \frac{s}{\ell}v_m$ (v_m = snelheid massa m) $\Rightarrow dK = \frac{1}{2}\left(\frac{M}{\ell}ds\right)\left(\frac{s}{\ell}v_m\right)^2$

$\Rightarrow K_{veer} = \int_0^\ell \frac{1}{2}\frac{M}{\ell}ds\left(\frac{s}{\ell}v_m\right)^2 = \frac{1}{2}\frac{Mv_m^2}{\ell^3}\left[\frac{1}{3}s^3\right]_0^\ell = \frac{1}{6}Mv_m^2$

Totale energie $E = \frac{1}{2}mv_m^2 + \frac{1}{6}Mv_m^2 + \frac{1}{2}kx_m^2 \Rightarrow \omega_0^2 = \frac{k}{m + \frac{1}{3}M} = \frac{75}{25.1}$ Hz^2

$\omega_0 = 1.729$ Hz

e. De aanname $v(s) \sim s$ geldt dan niet, er zouden lopende golven in de veer ontstaan

2a.  $v = i_2 z_2$ ① $\quad dv = -i_1 z_1 dx$ ② $\quad di_1 = -i_2 dx$ ③ $\quad$ (allen $\sim e^{i\omega t}$)

② $\Rightarrow \frac{d^2v}{dx^2} = -z_1\frac{di_1}{dx} = z_1 i_2 = \frac{z_1}{z_2}v = \gamma^2 v$ ④ met $\gamma = \sqrt{\frac{z_1}{z_2}}$

Oplossing ④ $v = Ce^{-\gamma x} + De^{\gamma x}$ Hier $x \to \infty$, eis v begrensd $\Rightarrow D = 0$

$\Rightarrow v = Ce^{-\gamma x}$ $\quad$ ② $\Rightarrow i_1 = -\frac{1}{z_1}\frac{dv}{dx} = C\frac{\gamma}{z_1}e^{-\gamma x}$ $\quad i_1\big|_{x=0} = I_0 = C\frac{\gamma}{z_1}$

$\Rightarrow C = \frac{z_1}{\gamma}I_0 \Rightarrow v = I_0\frac{z_1}{\gamma}e^{-\gamma x} = I_0\sqrt{z_1 z_2}\,e^{-\gamma x}$

$\Rightarrow V(x,t) = I_0 Z_0 e^{-\gamma x}e^{i\omega t}$ met $Z_0 = \sqrt{z_1 z_2}$ $\quad \left(Z_0 = Z_{in}\big|_{x=0} = \frac{v}{I_0}\big|_{x=0} = \sqrt{z_1 z_2}\right)$

b. $\gamma = \sqrt{\frac{z_1}{z_2}} = \sqrt{\frac{i\omega L}{R}} = \sqrt{\omega}\sqrt{\frac{L}{R}}\sqrt{i} = \sqrt{\omega}\sqrt{\frac{L}{R}}\left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}} = \sqrt{\omega}\sqrt{\frac{L}{R}}\frac{1}{\sqrt{2}}(1+i) = \sqrt{\omega}\sqrt{\frac{L}{2R}} + i\sqrt{\omega}\sqrt{\frac{L}{2R}}$

$v \sim e^{-\gamma x} = e^{-\mathrm{Re}(\gamma)x}e^{-i\,\mathrm{Im}(\gamma)x}$, dus $\mathrm{Re}(\gamma)$ verzwakking, $\mathrm{Im}(\gamma) = k$ = golfgetal

c. $k = \sqrt{\frac{L}{2R}}\,\omega^{\frac{1}{2}} \Rightarrow \omega = \frac{2R}{L}k^2 \Rightarrow v_{fase} = \frac{\omega}{k} = \frac{2R}{L}k$

$v_{groep} = \frac{d\omega}{dk} = \frac{2R}{L}2k = 2\,v_{fase}$

d. 1) pulsdispersie (zie c)

2) extreme demping $\sim e^{-\mathrm{Re}(\gamma)x}$; hier is $\mathrm{Re}(\gamma) = \mathrm{Im}(\gamma) = k \Rightarrow$ demping $\sim e^{-kx}$

dus amplitude $\frac{1}{e}$ gereduceerd voor $\Delta x = \frac{1}{k} = \frac{\lambda}{2\pi} \sim$ golflengte

3a  $E + E' = E''$ ① $H \sim kE \quad H'' \sim k''E''$

$-H\cos\theta + H'\cos\theta = -H''\cos\varphi \Rightarrow -kE\cos\theta + k'E'\cos\theta = -k''E''\cos\varphi$ ②

① $\rightarrow$ $E' = E'' - E \overset{②}{=} \dfrac{-kE\cos\theta + k'E'\cos\theta}{-k''\cos\varphi} - E$ ③ $k = \frac{\omega}{c}n_1 \quad k' = \frac{\omega}{c}n_1 \quad k'' = \frac{\omega}{c}n_2$

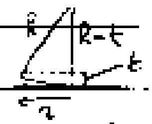
③ $\rightarrow E'\left[1 + \frac{k'}{k''}\frac{\cos\theta}{\cos\varphi}\right] = E'\left[1 + \frac{n_1}{n_2}\frac{\cos\theta}{\cos\varphi}\right] = E\left[\frac{n_1}{n_2}\frac{\cos\theta}{\cos\varphi} - 1\right] \Rightarrow r_s = \frac{E'}{E} =$

$= \dfrac{n_1\cos\theta - n_2\cos\varphi}{n_2\cos\varphi + n_1\cos\theta} = \dfrac{\cos\theta - n\cos\varphi}{\cos\theta + n\cos\varphi}$ met $n = n_2/n_1$

b. $\theta = \varphi = 0 \rightarrow r_s = r_p = (1-n)/(1+n) < 0$ voor $n > 1$ $\Rightarrow$ fasedraaiing $180°$

c. $|r_s|^2 = |r_p|^2 = \left(\frac{1-n}{1+n}\right)^2 = \left(\frac{n_1 - n_2}{n_1+n_2}\right)^2$ lucht $\rightarrow$ glas $n_1 = 1 \; n_2 = 1.7 \rightarrow \frac{I_2}{I_1} = |r|^2 = \left(\frac{0.7}{2.7}\right)^2 = 0.067$

kwarts $\rightarrow$ lucht $n_1 = 1.45 \quad n_2 = 1 \rightarrow R = \frac{I_2}{I_1} = 0.034$

d.  $R^2 = r^2 + (R-t)^2 = r^2 + R^2 - 2Rt + t^2 \approx r^2 + R^2 - 2Rt$ als $t \ll R$

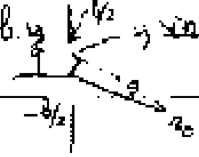
$\rightarrow t = \dfrac{r^2}{2R}$

Constructieve interferentie: $2t = (m + \frac{1}{2})\lambda = \frac{r^2}{R}$ (m geheel getal, $\frac{1}{2}$ komt van fasesprong bij overgang lucht $\rightarrow$ optisch vlak)

e. 1e lichtstraal: reflectie kwarts-lucht $\rightarrow R = \frac{I_2}{I_1} = 0.034 \rightarrow$ doorgelaten $T = 1 - R = 0.966$

2e lichtstraal: reflectie van glas $R = 0.067$, doorgelaten door lucht-kwarts op terugweg weer $T = 0.966 \rightarrow R_2 = 0.966 \times 0.067 \times 0.966 = 0.063$, dus bijna twee maal zoveel als voor 1e lichtstraal. Gevolg: minder diepe modulatie interferentiepatroon

4a. r' constant, r varieert weinig, maar $ikr = i2\pi\frac{r}{\lambda}$ wel. P niet te ver naast hoofdas $\rightarrow \cos(\vec{n},\vec{r}) \approx 1 \rightarrow$ alleen e^{ikr} binnen integraal houden: $U_p = C\iint e^{ikr}dA$

b.  $U = C\int_{-b/2}^{b/2} e^{ik(r_0 + y\sin\theta)}\,dy\,L = C'\int_{-b/2}^{b/2} e^{iky\sin\theta}\,dy = C'\frac{1}{ik\sin\theta}\left[e^{iky\sin\theta}\right]_{-b/2}^{b/2} =$

$= 2C'\frac{1}{k\sin\theta}\sin\left(\frac{1}{2}kb\sin\theta\right) \rightarrow I = |U|^2 = 4C'^2\frac{1}{k^2\sin^2\theta}\sin^2\left(\frac{1}{2}kb\sin\theta\right)$

$I = 0$ als $\frac{1}{2}kb\sin\theta = \pi \rightarrow \sin\theta = \frac{\lambda}{b}$ 

c. $\begin{matrix} b/2 \\ a/2 \\ -a/2 \\ -b/2 \end{matrix}$ $U = CLe^{ikr_0}\left\{\int_{-b/2}^{-a/2} e^{iky\sin\theta}\,dy + \int_{a/2}^{b/2} e^{iky\sin\theta}\,dy\right\} = \dfrac{CLe^{ikr_0}}{ik\sin\theta}\left\{e^{-ik\frac{a}{2}\sin\theta} - e^{-ik\frac{b}{2}\sin\theta} + e^{ik\frac{b}{2}\sin\theta} - e^{ik\frac{a}{2}\sin\theta}\right\} = C''\frac{1}{k\sin\theta}\left\{\sin\left(\frac{1}{2}kb\sin\theta\right) - \sin\left(\frac{1}{2}ka\sin\theta\right)\right\}$

d. Diffractiepatroon spleet met breedte a: $U_a = 2CLe^{ikr_0}\frac{1}{k\sin\theta}\sin\left(\frac{1}{2}ka\sin\theta\right)$

Babinet $\rightarrow$ diffractiepatroon strip met breedte a: $U_s = -U_a$

Superpositie: spleet met breedte b + strip: $U_{totaal} = U_b + U_s = U_b - U_a =$

$= 2 C_L e^{[\text{illegible}]} \left\{ \frac{1}{k \sin\theta} \sin\left(\frac{1}{2} k b \sin\theta\right) - \frac{1}{k \sin\theta} \sin\left(\frac{1}{2} k a \sin\theta\right) \right\} \doteq$

$\stackrel{!}{=}$ resultaat van c